AHMAD IBRAHIM SECONDARY SCHOOL
END-OF-YEAR EXAMINATION 2023

SECONDARY 3 EXPRESS

Name: MARKING SCHEME	Class:	Register No.:

ADDITIONAL MATHEMATICS **4049**
10 October 2023
Candidates answer on the Question Paper. **2 hours 15 minutes**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use
/ 90

This document consists of **17** printed pages.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 Dan invested \$$x$ in a bank at the start of 2023. The amount of money, \$$P$, at the end of t years is denoted by $P = 6800(1.015)^t$.

(a) State the value of x. **[1]**

When $t = 0$, $P = x = 6800$

B1

(b) Find the year in which the amount of money in the bank first reach \$8000. **[2]**

$$P = 6800(1.015)^t = 8000$$
$$1.015^t = \frac{20}{17}$$
$$t \ln 1.015 = \ln\frac{20}{17}$$
$$t = \frac{\ln\frac{20}{17}}{\ln 1.015}$$

M1

$$t = 10.916 \text{ (5 s.f.)}$$

Year required: 2033

A1

2 Without using a calculator, find the value of a and of b such that $\sqrt{a - b\sqrt{6}} = \frac{2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}}$. **[4]**

$$a - b\sqrt{6} = \left(\frac{2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}}\right)^2$$
$$= \frac{4(3)}{9(2) + 4(3) + 12\sqrt{6}}$$
$$= \frac{12}{30 + 12\sqrt{6}}$$
$$= \frac{2}{5 + 2\sqrt{6}} \times \frac{5 - 2\sqrt{6}}{5 - 2\sqrt{6}}$$
$$= \frac{10 - 4\sqrt{6}}{25 - 4(6)}$$
$$= 10 - 4\sqrt{6}$$

$a = 10$, $b = 4$

M1: for correct expansion or show

$\sqrt{a - b\sqrt{6}} = \frac{2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}} = \sqrt{6} - 2$ (rationalize)

M1: for correctly rationalising their expression or

Show $\left(\sqrt{6} - 2\right)^2 = 10 - 4\sqrt{2}$

A1 for each correct answer

3 Express $\frac{13x - 6}{x^2(2x - 3)}$ in partial fractions. **[5]**

$$\frac{13x - 6}{x^2(2x - 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x - 3}$$
$$13x - 6 = Ax(2x - 3) + B(2x - 3) + Cx^2$$

When $x = 0$: $-6 = -3B \Rightarrow B = 2$

When $x = \frac{3}{2}$: $\frac{27}{2} = \frac{9}{4}C \Rightarrow C = 6$

When $x = 1$: $7 = -A - 2 + 6 \Rightarrow A = -3$

$$\frac{13x - 6}{x^2(2x - 3)} = \frac{2}{x^2} - \frac{3}{x} + \frac{6}{2x - 3}$$

M1: correct form identified

M1 for setting up 3 correct eqns for solving 3 unknowns (comparing coeff) OR for substituting logical values to find unknowns

A1: any 1 correct constant

A2: any 2 correct constants

A3: all correct constants and writing out the partial fraction decomposition correctly. Not awarded if students did not write the final line.

4 A golfer hits a ball such that the height, h metres, of the ball above the ground t seconds later is given by $h = 0.2t(40-t)$.

(a) By expressing the function in the form $h = a(t-m)^2 + n$, where a, m and n are constants, explain whether the ball can reach a height of 100 metres. **[2]**

$$h = 0.2t(40-t)$$
$$= -0.2(t^2 - 40t)$$
$$= -0.2\left[(t-20)^2 - 400\right]$$
$$= -0.2(t-20)^2 + 80$$

M1: correct completing the square

Since the maximum point of the ball's trajectory is at 80m, it will not reach a height of 100m.

A1: correct conclusion based on max point

Or

Use inequality to show h is below or equal to 80

$(t-20)^2 \geq 0$
$-0.2\,(t-20)^2 \leq 0$
$-0.2\,(t-20)^2 + 80 \leq 80$

(b) Find the range of values of t for which the ball is at most 75 metres above the ground. **[4]**

$0.2t(40-t) \leq 75$
$40t - t^2 \leq 375$
$t^2 - 40t + 375 \geq 0$
$(t-25)(t-15) \geq 0$
$t \leq 15$ or $t \geq 25$

Since $0 \leq t \leq 40$, $0 \leq t \leq 15$ or $25 \leq t \leq 40$

M1: correct quadratic inequality set up

If use ≥ no M1 but award ecf 1 for subsequent steps

M1: factorising or graph sketching or formula to solve

M1: obtaining correct answer

A1: took into account range of t to reach final answer

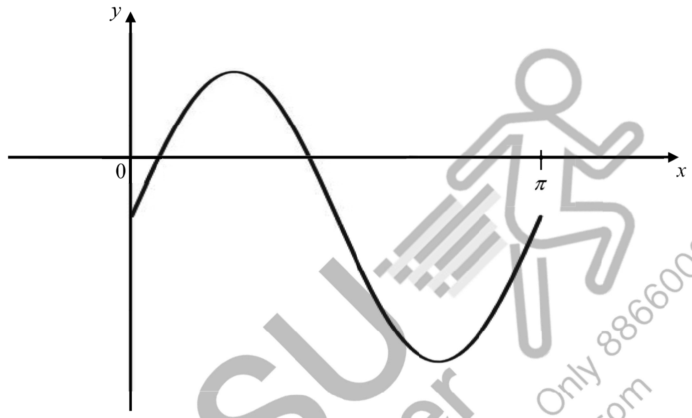
Note: if students solve equation $0.2t(40-t) = 75$ instead and give correct final range, award full credit.
If solve equation but wrong range, max 1m.

5(a) Write down the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ in radian as a multiple of π. **[1]**

Principal value of $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

B1

(b)



The diagram shows the curve $y = a\sin bx + c$ for $0 \leq x \leq \pi$ radians. The curve has a maximum point at $\left(\frac{\pi}{4}, 3\right)$ and a minimum point at $\left(\frac{3\pi}{4}, -7\right)$.

(i) Explain why $c = -2$. **[2]**

Or use amplitude (show that it is 5) to show shifting in relation to the max or min value to get c = – 2

Since min $y = -7$ and max $y = 3$, **<u>centreline</u>** $y = c$ at $y = \frac{3+(-7)}{2} = -2$. Therefore $c = -2$.

B1: keyword, can use midline

B1: calculation

(ii) Explain why $b = 2$. **[2]**

<u>Period</u> of $y = a\sin bx + c$ graph $= \frac{2\pi}{b} = \pi$ from graph. Therefore, $b = 2$

B1: keyword

Or use 'duration of one cycle'

B1: calculation

Either in using π or deg

(iii) Hence find the equation of the curve. **[2]**

Amplitude $= \frac{3-(-7)}{2} = 5$

B1: correct amplitude

Eqn of graph: $y = 5\sin 2x - 2$

B1: must see y=

6(a) The curve with equation $y = ax^2 + bx + a$, where a and b are constants, lies completely above the x-axis.

(i) Write down the conditions which must apply to a and b. **[2]**

$a > 0$ and $b^2 - 4a^2 < 0$

$-2a < b < 2a$

B1: $a > 0$
B1: $-2a < b < 2a$

(ii) Give an example of possible values for a and b which satisfy the conditions in part **(i)**. **[2]**

Any positive value for a and $-2a < b < 2a$

B1: any $a > 0$
B1: any $b^2 - 4a^2 < 0$ with their selected value of a

(b) Using a suitable substitution, explain why the equation $3^{2x-1} = 3^{x+2} - k$ has no solution if $k > 60.75$. **[3]**

$3^{2x-1} = 3^{x+2} - k$

$\frac{1}{3}(3^x)^2 - 3^2(3^x) + k = 0$

Let $u = 3^x$

$\frac{1}{3}u^2 - 9u + k = 0$

M1: Applying indices laws to write into quadratic equation

$u^2 - 27u + 3k = 0$

Discriminant $= (-27)^2 - 4(3k) = 729 - 12k$

When $k > 60.75$, $-12k < -729$

$729 - 12k < 0$

M1: correct discriminant

can show $81 - \frac{4}{3}k$

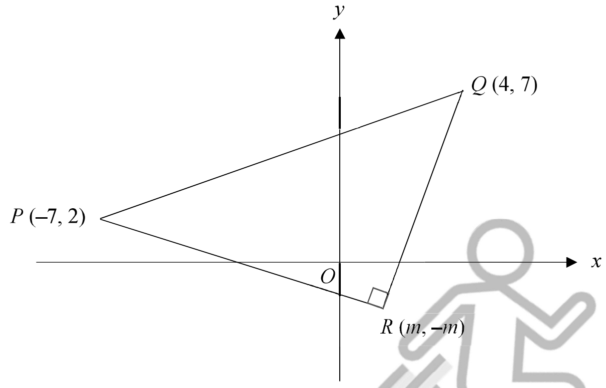
Since discriminant < 0 when $k > 60.75$, the equation $u^2 - 27u + 3k = 0$ has no real solution. Thus $3^{2x-1} = 3^{x+2} - k$ has no solution.

A1: showing discriminant <0 when $k > 60.75$ and giving conclusion

or state assumption no solution, so discriminant < 0 and then show that $k > 60.75$

or use completing square to show min point (13.5, k − 60.75) and explain the quadratic curve is above the zero-value line i.e. no intersection, no solution

7



The diagram shows a triangle PQR and angle PRQ is 90°.
The coordinates of P, Q and R are $(-7, 2)$, $(4, 7)$ and $(m, -m)$ respectively, where m is a constant.

(a) Show that $m = 1$. **[2]**

Alternatively, use Pythagoras' thm to form equation, get m = 1 or m = - 7 *reject)

Gradient of $PR = \frac{2+m}{-7-m} = -\frac{2+m}{7+m}$

Gradient of $QR = \frac{7+m}{4-m}$

B1: either 1 gradient correct

$-\frac{2+m}{7+m} = -\frac{4-m}{7+m}$

B1: correct equation and reaching m=1

$2 + m = 4 - m$

$m = 1$

Alternatively, use gradient of PQ × gradient of QR = – 1
Show (m + 7)(m – 1) = 0, get m = 1 or m = - 7 (reject)

(b) $PQRS$ is a parallelogram. Find the coordinates of S. **[2]**

Let coordinate of S be (x, y).

Midpoint of QS = midpoint of PR

$\left(\frac{4+x}{2}, \frac{7+y}{2}\right) = \left(\frac{-7+1}{2}, \frac{2-1}{2}\right)$ M1

$x = -10$, $y = -6$

Coor of S = $(-10, -6)$ A1

(c) T is the point on PR such that $PT : TR = 1 : 3$.
Find the coordinates of T. **[2]**

Let coordinate of T be (a, b).

Using similar triangles, $a = -7 + \dfrac{1+7}{4} = -5$

$b = 2 - \dfrac{2+1}{4} = 1.25$

coordinate of T be (-5, 1.25)

M1: any sound method (e.g. similar triangles, counting of units, finding length)

A1

(d) Find the area of the triangle PTQ. **[2]**

$$\text{Area} = \frac{1}{2}\begin{vmatrix} -7 & -5 & 4 & -7 \\ 2 & 1.25 & 7 & 2 \end{vmatrix}$$

$$= \frac{1}{2}(-8.75 - 35 + 8 + 10 - 5 + 49)$$

$$= 9.125 \text{ units}^2$$

M1: if using shoelace, must be anti-clockwise, or can use 0.5xQRx0.25xPR. Allow ECF.

A1, accept $\dfrac{73}{8}$ units^2

OR

$$PR = \sqrt{(-7-1)^2 + (2+1)^2} = \sqrt{73}$$

$$QR = \sqrt{(4-1)^2 + (7+1)^2} = \sqrt{73}$$

$$\text{Area} = \frac{1}{2} \times PT \times QR$$

$$= \frac{1}{2} \times \frac{1}{4} PR \times QR$$

$$= \frac{1}{2} \times \frac{1}{4}\sqrt{73} \times \sqrt{73} = \frac{73}{8}$$

M1: 0.5xQRx0.25xPR. Allow ECF.

8(a) Solve the equation $\log_2(x^2 + 4x - 1) = 2 + 2\log_4(3x+2)$. **[5]**

$$\log_2(x^2 + 4x - 1) = 2 + 2\frac{\log_2(3x+2)}{\log_2 4}$$

M1 for correct change of base

$$\log_2(x^2 + 4x - 1) = 2 + \log_2(3x+2)$$

$$\log_2\frac{(x^2 + 4x - 1)}{(3x+2)} = 2$$

M1 for applying laws of logarithm to arrive at quadratic eqn

$$\frac{(x^2 + 4x - 1)}{(3x+2)} = 4$$

$$x^2 + 4x - 1 = 12x + 8$$

M1 correct quadratic eqn

$$x^2 - 8x - 9 = 0$$

$$x = 9 \quad \text{or} \quad x = -1$$

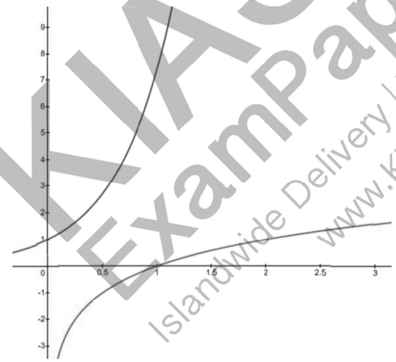
M1 for correctly solving their quadratic

When $x = -1$, $\log_2(x^2 + 4x - 1)$ and $\log_4(3x+2)$ are undefined.

Therefore $x = 9$

A1 for correct answer with proper justification/check

(b) Sketch the graphs of $y = \log_2 x$ and $y = e^{2x}$ on the same axes. Use your graphs to explain why $\log_2 x + e^{2x} > 0$ for $x > 1$. **[4]**



B1 for $y = e^{2x}$ with correct y-int

B1 for $y = \log_2 x$ with correct x-int

-1 overall if graphs intersect, or graphs did not tend towards axis

When $x > 1$, $\log_2 x > 0$ and $e^{2x} > e^2 > 0$.

Since both graphs/functions are always increasing,

$\log_2 x + e^{2x} > 0$ for $x > 1$.

B1: $\log_2 x > 0$ and $e^{2x} > e^2 > 0$
B1: always increasing, and reaching conclusion

9 The polynomial $g(x)=mx^3-8x^2-9x+n$ has a factor of $3x-2$ and leaves a remainder of 6 when divided by x.

(a) Find the values of m and n. **[3]**

Factor of $3x-2$: $g\left(\frac{2}{3}\right)=m\left(\frac{2}{3}\right)^3-8\left(\frac{2}{3}\right)^2-9\left(\frac{2}{3}\right)+n=0$

$$\frac{8}{27}m+n=\frac{86}{9}$$

M1: subst x=2/3, remainder = 0

Remainder of 6 when divided by x: $g(0)=m(0)^3-8(0)^2-9(0)+n=6$ (0)

$n=6$ A1

$m=12$ A1

(b) Using the values of m and n from **(a)**, solve, in exact form, the equation $g(x)=0$. **[3]**

$g(x)=12x^3-8x^2-9x+6$

$=(3x-2)(4x^2-3)=0$

M1 for correct factors

$3x-2=0$ or $4x^2-3=0$

$x=\frac{2}{3}$ or $x=\pm\frac{\sqrt{3}}{2}$

A2: for all correct exact answers (accept $x=\pm\sqrt{\frac{3}{4}}$)

A1: any 2 correct

(c) Hence solve the equation $9e^{2y}+8e^{y}=12+6e^{3y}$. **[3]**

$9e^{2y}+8e^{y}=12+6e^{3y}$

$12-8e^{y}-9e^{2y}+6e^{3y}=0$

$12e^{-3y}-8e^{-2y}-9e^{-y}+6=0$

M1 for correct substitution

$e^{-y}=\frac{2}{3}$ or $e^{-y}=\pm\frac{\sqrt{3}}{2}$ (reject $-\frac{\sqrt{3}}{2}$ as $e^{-y}>0$ for all real values of y)

$y=-\ln\left(\frac{2}{3}\right)$ or $y=-\ln\left(\frac{\sqrt{3}}{2}\right)$

M1 for rejecting and taking ln of 2 terms

$=0.405$ or $y=0.144$

A1 for each answers

10 A tangent to a circle at the point $(3,1)$ intersects the x-axis at $x=5$.

(a) Find the equation of the tangent T. **[2]**

Gradient of tangent $=\frac{1-0}{3-5}=-\frac{1}{2}$

M1 for correct gradient

Eqn of tangent:

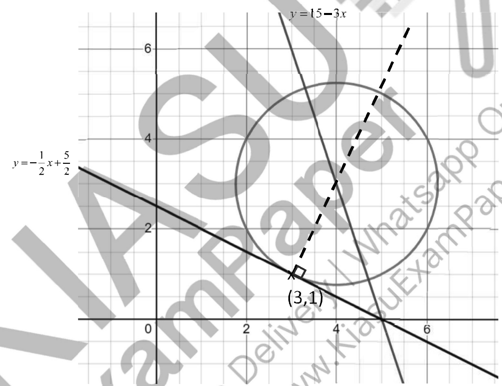
$y-0=-\frac{1}{2}(x-5)$

$y=-\frac{1}{2}x+\frac{5}{2}$

A1

(b) The centre of the circle lies on the line $y=15-3x$.

Find the equation of the circle. **[6]**



Let centre of the circle be C $(x, 15-3x)$

Let P denote the point on circle (3,1).

Gradient of CP $=\frac{15-3x-1}{x-3}$

M1: finding expression for gradient of perpendiular

$=\frac{14-3x}{x-3}$

$=2$ (tangent perpendicular to radius)

M1: their grad of normal

$$14-3x=2x-6$$
$$5x=20$$
$$x=4$$
$$y=15-3x=3$$

M1: equating the gradient to their gradient of normal

Centre: (4, 3)

M1: correct centre

Radius $=\sqrt{(4-3)^2+(3-1)^2}=\sqrt{5}$

M1: finding radius from their centre

Eqn: $(x-4)^2+(y-3)^2=5$

A1

OR

Gradient of normal = 2

M1: their grad of normal

Equation of normal: $y-1=2(x-3)$

$y=2x-5$

M1: correct eqn of normal

At centre:

$2x-5=15-3x$

$x=4$

$y=3$

M1: intersection of their tangent and normal

Centre: (4, 3)

M1: correct centre

Radius $=\sqrt{(4-3)^2+(3-1)^2}=\sqrt{5}$

M1: finding radius from their centre

Eqn: $(x-4)^2+(y-3)^2=5$

A1

(c) Find the equations of the tangents to the circle parallel to the y-axis. **[2]**

Parallel to $y-$ axis: gradient is undefined (vertical lines)

Eqn of tangent: $x=4\pm\sqrt{5}$

B2 for both correct answers, B1 for any one

11 In the expansion of $\left(x^2+\frac{1}{3x}\right)^n$ in descending powers of x, the coefficients of the second and the third term are in the ratio of $6:11$.

(a) Show that the value of n is 12. **[6]**

$$\left(x^2+\frac{1}{3x}\right)^n=\left(x^2\right)^n+\binom{n}{1}\left(x^2\right)^{n-1}\left(\frac{1}{3x}\right)+\binom{n}{2}\left(x^2\right)^{n-2}\left(\frac{1}{3x}\right)^2+\ldots$$

M1: correct appln of formula

$$=\left(x^{2n}\right)+n\left(x^{2n-2}\right)\left(\frac{1}{3}\right)x^{-1}+\frac{n(n-1)}{2}\left(x^{2n-4}\right)\left(\frac{1}{3}\right)^2x^{-2}+\ldots$$

M1: correct evaluation of binomial coefficient

$$=x^{2n}+\left(\frac{1}{3}\right)nx^{2n-3}+\frac{n(n-1)}{18}x^{2n-6}+\ldots$$

M1 each: correct 2nd and 3rd term

Coefficient of 2nd term : coefficient of 3rd term $=\frac{1}{3}n:\frac{n(n-1)}{18}$

$$\frac{\frac{1}{3}n}{\frac{n(n-1)}{18}}=\frac{6}{11}$$

M1: correct set up of eqn

Since $n\neq 0$, $\frac{6}{n-1}=\frac{6}{11}$

$n-1=11$

$n=12$

A1

OR

General term $=\binom{n}{r}\left(x^2\right)^{n-r}\left(\frac{1}{3x}\right)^r$

M1: correct appln of formula

$$=\binom{n}{r}\left(3^{-r}\right)\left(x^{2n-3r}\right)$$

M1: correct evaluation of binomial coefficient

M1: Subst n = 1 and n = 2 for 2nd and 3rd term respectively

2nd term: $\left(\frac{1}{3}\right)nx^{2n-3}$

3rd term: $\frac{n(n-1)}{18}x^{2n-6}$

M1 each for correct 2nd and 3rd term

(b) Find the middle term. **[2]**

Total terms = 13

Middle term = 7th Term

$T_{r+1} = T_7$
$r = 6$
Middle term
$\binom{12}{6}(x^2)^{12-6}\left(\frac{1}{3x}\right)^6$
$\therefore \frac{308}{243}x^6$

M1: identifying r=6

A1

(c) Find the coefficient of x^6 in the expansion of $(2-x^3)\left(x^2+\frac{1}{3x}\right)^{12}$. **[3]**

$\left(x^2+\frac{1}{3x}\right)^{12}$: General term = $\binom{12}{r}(3^{-r})(x^{24-3r})$

x^6 term: $r = 6$
x^3 term: $r = 7$

M1: finding the necessary r-values / expansion

coefficient of x^6 in $(2-x^3)\left(x^2+\frac{1}{3x}\right)^{12} = (2)\left(\frac{308}{243}\right) - 1\binom{12}{7}(3^{-7}) = \frac{176}{81}$

M1, A1

12(a) Variables x and y are related by the equation $yx^n = k$, where n and k are constants. Explain clearly how a straight line graph can be drawn to represent the relationship, and state how the values of n and k could be obtained from the line. **[4]**

$yx^n = k$
$\ln(yx^n) = \ln k$
$\ln y + n\ln x = \ln k$
$\ln y = -n\ln x + \ln k$

M1: taking ln or lg

M1: correctly rewriting the straight line form

By plotting $\ln y$ against $\ln x$, a straight line with gradient = $-n$ and y-intercept = $\ln k$ will be obtained.

B1: correctly identifying what to plot against

$n = -$gradient of line, $k = e^{y\text{-int}}$

B1: state how the values of n and k could be obtained from the line

(b) The time for a complete oscillation, t seconds, of a pendulum of length l m is proportional to $\sqrt{l}$. In an experiment with pendulums of different lengths, the following table was obtained.

Length of pendulum, l m	0.2	0.4	0.6	1.0
Time of one oscillation, t sec	0.90	1.27	1.55	2.02

(i) On the grid on **page 17**, draw a straight line graph to illustrate this data. **[2]**

(ii) Use your graph to estimate the time of one oscillation for a pendulum of length 0.8 m. **[2]**

$l = 0.8$
$\sqrt{l} = 0.894$
From graph, $t = 1.8s$ (allow ± 0.1)

M1: find $\sqrt{l}$ and read off graph
A1

(iii) It is known that the correct formula connecting t and l is $t = 2\pi\sqrt{\frac{l}{g}}$, where g is the acceleration due to gravity.
Use your graph to estimate the value for g. **[3]**

$$\text{gradient} = \frac{2\pi}{\sqrt{g}}$$
$$= \frac{1.8-0}{0.89-0}$$
$$= 2.0225 \text{ (allow 1.95 - 2.05)}$$

M1: realised grad = 2pi/rt(g)

M1: finding gradient from their graph

$$g = \left(\frac{2\pi}{2.0335}\right)^2 = 9.65 \text{ (allow 9.4 to 10.3)}$$

A1

l	0.2	0.4	0.6	1.0
t	0.90	1.27	1.55	2.02
$\sqrt{l}$	0.447	0.632	0.775	1

END OF PAPER